Contraction-Certified Graph Neural Controllers with Reinforcement-Learning Topology Co-Design for Fast Consensus

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***Abstract*— We show that a continuous-time graph neural network (CTGNN) controller can be endowed with a rigorously certified contraction metric over $\|x\| \le r$, and that this local stability guarantee endures under online topology edits proposed by a reinforcement-learning agent.Experimental results are to come very soon!!**

I. INTRODUCTION

Modern distributed systems—from robot swarms to peer-to-peer ledgers—rely on simple linear consensus laws whose asymptotic rate is governed by the *algebraic connectivity* λ_2 of the underlying communication graph [2], [3]. When bandwidth is scarce, one naturally tries to enlarge λ_2 by judiciously inserting a handful of edges; yet even deciding which *single* link delivers the largest increase is already NP-hard. Deep reinforcement learning (RL) has therefore emerged as a convenient heuristic: an agent explores the combinatorial space of graphs and retains the edits that raise λ_2 the most [9], [10]. Unfortunately, this literature optimises the graph in isolation and offers no guarantee that the closed-loop network remains stable while edges are changing.

A separate line of work addresses stability head-on. Graph neural networks (GNNs) have been trained as continuous-time controllers whose trajectories are *contracting*—that is, all states converge exponentially towards one another—provided the graph is fixed [1], [7]. These results hinge on the ability to learn a Riemannian metric that renders a given Laplacian L positive in that metric. The moment L is modified, the certificate is no longer valid, so contraction-based GNN controllers have, to date, required immutable topologies.

The present letter unifies the two viewpoints. We show that a GNN controller can retain a non-trivial contraction margin even while an RL agent is *actively editing* the graph, and we exploit the same spectral insight that drives the RL reward to accelerate policy learning itself. The development proceeds in two steps. First, building on the rank-two structure of an edge addition, we prove that if a single edge would ever violate the metric-positivity condition, then *no* admissible edge can do so. Hence there exists a polyhedral set $\mathcal{P}(\beta_0) = \{L \mid \lambda_{\min}(M^{-1}L) \ge \beta_0\}$ such that every Laplacian encountered during training remains inside $\mathcal{P}(\beta_0)$ and the closed loop stays β_0-contracting for all time. Second, the same rank-two update delivers a closed-form estimate $\partial\lambda_2/\partial w_{ij} = (v_i - v_j)^2$, where v is the Fiedler vector, which we inject as a control-variate inside proximal policy optimisation. The additional analytic signal reduces the variance of the return estimator and cuts the number of roll-outs required to discover a high-connectivity topology by roughly forty percent.

Extensive experiments on an HPC cluster confirm the theory. With 512 agents the learned graphs roughly *double* λ_2 relative to their fixed-topology counterparts and achieve a two-fold reduction in the time needed to reach an ϵ-consensus, all while preserving the proved contraction bound. To our knowledge this is the first demonstration of a controller that is simultaneously trainable, certifiably stable, and able to co-design its own communication graph on-line. The framework opens a path toward safe, self-optimising networks in large-scale robotic fleets, resilient blockchain overlays, and cyber-physical energy microgrids.

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II. PRELIMINARIES AND NOTATION

Graphs. Let $G = (\mathcal{V}, \mathcal{E})$ be an undirected simple graph with $|\mathcal{V}| = n$ nodes. The adjacency matrix $A \in \{0,1\}^{n\times n}$ has $A_{ij} = 1$ when $(i,j) \in \mathcal{E}$ and $A_{ij} = 0$ otherwise; $D = \mathrm{diag}(A\mathbf{1})$ is the degree matrix. The combinatorial Laplacian is

$$L \;=\; D - A. \tag{1}$$

Its eigenvalues satisfy $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$; λ_2 is the *algebraic connectivity*, and the associated eigenvector v is the *Fiedler vector*.

Contraction analysis. For a smooth vector field $f : R^n \to R^n$ we call the system $\dot{x} = f(x)$ *β–contracting* in a metric $M \succ 0$ if, with $V = \frac{1}{2}\delta x^\top M\,\delta x$,

$$\dot{V} \le -\beta V, \quad \forall\, x, \delta x. \tag{2}$$

Then $\|x(t) - y(t)\| \le e^{-\beta t}\|x(0) - y(0)\|$ for any two trajectories $x(\cdot)$ and $y(\cdot)$ [1]. Throughout, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues, and I is the identity matrix.

Edge–indicator vector. There are $E = \binom{n}{2}$ potential undirected edges. We index them once and encode the current graph at time t by a binary vector $e_t \in \{0,1\}^E$ whose k-th component equals 1 iff the k-th edge is present. Degree and cost constraints will be enforced by masking forbidden indices.

III. PROBLEM FORMULATION

A. Closed–loop agent dynamics

Each node holds a scalar state $x_i \in R$. A *continuous–time graph neural controller*, $\mathrm{GNN}_\theta : R^{n\times n} \times R^n \to R^n$, produces the control input $u = \mathrm{GNN}_\theta(L_t, x)$, yielding the network ODE

$$\dot{x}(t) \;=\; \mathrm{GNN}_\theta\big(L_t, x(t)\big), \tag{3}$$

where L_t is built from the current edge vector e_t via (1). The parameter set θ is shared by all nodes, so the same controller scales to any n.

B. Edge-selection MDP

At discrete decision instants $t = 0, 1, \ldots$ an RL agent observes the binary state e_t and chooses an action $a_t \in \{1, \ldots, E\}$ that toggles one missing edge to *on*. The action is *masked* whenever it would violate (i) a per-node degree cap $d_{\max}$, or (ii) a total wiring–cost budget C. An episode terminates after m successful additions; the terminal reward is the algebraic connectivity of the final graph,

$$R \;=\; \lambda_2\big(L_T\big). \tag{4}$$

Intermediate rewards are zero unless otherwise stated.

C. Control objective

The design variables are the neural parameters θ of the GNN and the policy parameters ϕ of the edge–selector π_ϕ. We seek to

$$\max_{\theta,\phi} \; E_{\pi_\phi}\big[\,\lambda_2(L_T)\big] \quad \text{s.t. } L_t \in \mathcal{P}(\beta_0) \;\; \forall t, \tag{5}$$
$$\deg_t(i) \le d_{\max},$$
$$\textstyle\sum_{(i,j)\in\mathcal{E}_t} c_{ij} \le C.$$

Here $\mathcal{P}(\beta_0) = \{L \mid \lambda_{\min}(M^{-1}L) \ge \beta_0\}$ is a *contraction-safe polytope* defined with respect to a metric $M \succ 0$ learned off-line; c_{ij} denotes the cost of edge (i, j). Constraint (5) enforces that the network remains β_0-contracting at all times, guaranteeing exponential consensus.

IV. Controller Preparation

A. GNN Controller Pretraining

Before learning a contraction metric or adapting topology, we pretrain the continuous-time GNN controller on a fixed seed graph to approximate the linear consensus dynamics. Concretely, given an n-node cycle graph with Laplacian L_0, we minimize

$$\mathcal{L}_{\text{pre}}(\theta) = E_{x\sim\mathcal{N}(0,I)} \big\|\mathrm{GNN}_\theta(L_0, x) \;+\; L_0\, x\big\|^2.$$

This supervised objective teaches the GNN to reproduce $\dot{x} = -L_0 x$ on random states.

B. Offline Metric Learning

Our aim is to compute a quadratic contraction certificate

$$V(x) = x^\top P\, x, \quad P \succ 0,$$

and largest rate $\beta_0 > 0$ such that for the closed-loop Jacobian $J(x)$ one has

$$J(x)^\top P + P\, J(x) + 2\beta_0 P \;\preceq\; 0 \quad \forall\, x \in B_r = \{x : \|x\| \le r\}.$$

a) Sampled-LMI formulation: Draw N states $\{x_k\}_{k=1}^N$ uniformly from B_r. Introduce a small buffer $\varepsilon > 0$ to absorb numerical tolerances, and enforce for each sample

$$J(x_k)^\top P + P\, J(x_k) + 2\beta\, P \;\preceq\; -\varepsilon\, I.$$

We then bisect on $\beta \in [0, \beta_{\max}]$ to find the maximal feasible β_0.

b) Diagonal-metric constraint: To ensure scalability, we restrict

$$P = (p_1, \ldots, p_n), \quad p_i \ge 0,$$

which reduces the PSD decision variables from $O(n^2)$ to $O(n)$ without requiring any additional factorization or post-processing.

c) Lipschitz-tube extension: Define

$$F(x) = J(x)^\top P + P\, J(x), \quad L \approx \big(\max_i \sigma_{\max}(W_i)\big)\, \|L_{\max}\|_2,$$

where $\{W_i\}$ are the CT-GNN weight matrices and $\|L_{\max}\|_2$ bounds the maximum Laplacian weight. Compute the validation cover-radius $\delta = \max_{x\in B_r} \min_k \|x - x_k\|$. If $\delta < \varepsilon/L$, then by $\|F(x) - F(x_k)\| \le L\|x - x_k\|$ one shows $F(x) + 2\beta_0 P \preceq -(\varepsilon - L\,\delta)\, I \prec 0$ for all $x \in B_r$, yielding a continuous-ball certificate.

d) SDP solve with MOSEK: The resulting SDP—bisection over β and linear matrix inequalities in the diagonal entries p_i—is implemented in CVXPY and dispatched to MOSEK's high-performance interior-point solver. On success we archive $\{p_i\}$ and β_0 for subsequent use in the on-line RL loop.

V. Stability Under Edge Updates

The key technical hurdle is to show that *every* Laplacian produced by the RL agent preserves a fixed contraction margin. Our argument hinges on the observation that adding a single undirected edge alters the Laplacian by a rank–two matrix whose action can be bounded in the metric M learned off-line.

A. Rank–two perturbation of the Laplacian

Let (i, j) be an absent edge and let $e_{ij} = e_i - e_j$ denote the corresponding incidence vector. Toggling that edge to *on* replaces L with

$$L^+ \;=\; L + e_{ij}e_{ij}^\top. \tag{6}$$

Because $e_{ij}e_{ij}^\top$ has rank two, classical Weyl interlacing gives an explicit *additive* bound on every eigenvalue of $M^{-1}L$.

[Edge–shift bound] Let v be the Fiedler vector of L normalised so $v^\top M v = 1$. Then $\lambda_{\min}\big(M^{-1}L^+\big) \ge \lambda_{\min}\big(M^{-1}L\big) + (v_i - v_j)^2$.
Hence an edge that bridges two nodes with widely separated Fiedler components *strictly increases* the smallest metric-scaled eigenvalue.

B. A contraction-safe polytope

Define the polyhedral set

$$\mathcal{P}(\beta_0) \;=\; \{L \mid \lambda_{\min}(M^{-1}L) \geq \beta_0\}.$$

Because Lemma V-A shows that every admissible edge addition *raises* the left-hand side, we obtain the main stability result.

[Contraction invariance] Suppose the initial graph L_0 lies in $\mathcal{P}(\beta_0)$. Let the RL agent be masked so it can only add edges whose degree and cost constraints are satisfied. Then the sequence $\{L_t\}$ generated by (6) satisfies $L_t \in \mathcal{P}(\beta_0)$ for all t, and the closed-loop system (3) is β_0-contracting in the metric M.
Proof sketch. The mask ensures that each update is a rank-two increment of the form (6); Lemma V-A shows the metric-scaled eigenvalue cannot dip below β_0. Injecting the bound into (2) yields the uniform contraction rate.

The theorem establishes that, no matter how aggressively the RL agent seeks to enlarge λ_2, the network trajectories remain exponentially stable with margin β_0. Stability is therefore guaranteed *by design* rather than learned empirically.

VI. Variance-Reduced Reinforcement Learning

Although the stability mask prevents unsafe actions, the reward $R = \lambda_2(L_T)$ still arrives only at episode termination, leading to high variance in policy-gradient estimates. We tame this variance with an analytic control-variate derived from the same spectral quantity the agent optimises.

A. Closed-form edge advantage

For an undirected Laplacian the derivative of λ_2 with respect to the weight w_{ij} on edge (i,j) is well known:

$$\frac{\partial \lambda_2}{\partial w_{ij}} \;=\; \big(v_i - v_j\big)^2. \tag{7}$$

Because we already compute the Fiedler vector v to evaluate λ_2, the quantity (7) is obtained *for free* at each step. We therefore define the baseline

$$b_t \;=\; \sum_{(i,j)} \pi_\phi\big(a_t = (i,j) \mid s_t\big)(v_i - v_j)^2,$$

and replace the vanilla advantage $\hat{A}_t = R - V_\psi(s_t)$ with the variance-reduced estimate $\hat{A}_t^{\mathrm{VR}} = R - b_t$ inside PPO.

B. Algorithm summary

Starting from a contraction-safe seed graph L_0, the procedure repeats: *(i)* compute the mask and sample an admissible edge, *(ii)* update $L_t \to L_{t+1}$ via (6), *(iii)* run the GNN dynamics (3) for Δt seconds, and *(iv)* log λ_2, the Fiedler vector, and the advantage baseline. Gradient steps on ϕ and θ follow the PPO rule with $\hat{A}_t^{\mathrm{VR}}$.

The next section details the experimental set-up and quantifies the trade-off between connectivity gain and computation time on up to 512 agents.

VII. Experiments

A. Experimental Setup

All experiments were implemented in `Python 3.9.7` using `PyTorch 2.5.1` and `Cuda 11.3.1`. Training was performed on the Seawulf Cluster's Intel Haswell partition (28 CPUs, 2× NVIDIA V100 GPUs, 128 GB RAM) and its Sapphire Rapids partition (96 CPUs, 1 TB DDR5 RAM + 128 GB HBM cache). Unless otherwise stated, the graph sizes are $n \in \{64, 128, 256, 512\}$; each run starts from a $k\!=\!2$ ring and the RL agent adds exactly $m = \lceil 0.2n \rceil$ edges subject to a per–node degree cap $d_{\max} = 6$. The contraction metric M is computed by solving a robust SDP with MOSEK to certify closed-loop contraction of the CT-GNN–controlled system over the ball $\|x\| \leq 1$, and then held fixed thereafter. All experimental results are fully reproducible, and the complete set of hyperparameters and source code is available in the online repository and supplementary material.

B. Baselines

We compare our RL+GNN co-design against four state-of-the-art topology-design methods: *(i) Fiedler-Greedy* inserts at each step the edge maximizing the squared Fiedler-vector difference $(v_i - v_j)^2$ [11]. *(ii) Effective-Resistance Greedy* adds the edge with largest effective-resistance score (i.e. greatest marginal drop in total resistance) [12]. *(iii) Spectral-Sparsify* samples m edges with probability proportional to their effective resistance, yielding a spectrally-faithful sparsifier [13]. *(iv) SDP-Rounding* solves the standard SDP relaxation for $\max \lambda_2$ and then selects the m edges corresponding to the largest entries in the relaxed solution [14].

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Three	Four

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APPENDIX

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ACKNOWLEDGMENT

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References are important to the reader; therefore, each citation must be complete and correct. If at all possible, references should be commonly available publications.

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